Criticality and parallelism in combinatorial optimization

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Motivation

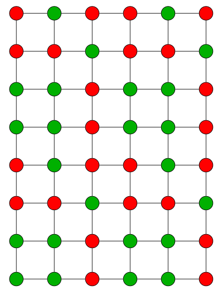
- Efficient techniques for tackling combinatorial optimization problems exploit the *structure* of the instance to attack
- Strong correlation between search effectiveness and some critical parameters of the instance (e.g., see studies on phase transitions)

Outline

Example: Optimizing on subsystems

Kauffman and Macready, Complexity 1995

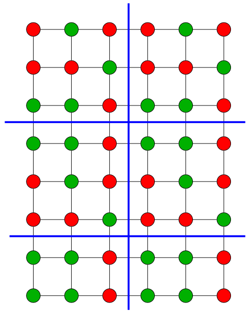
- Minimizing the energy of a spin glass system
- Total Energy = $\sum_i$ energy$_i$



Example: Optimizing on subsystems

Kauffman and Macready, Complexity 1995

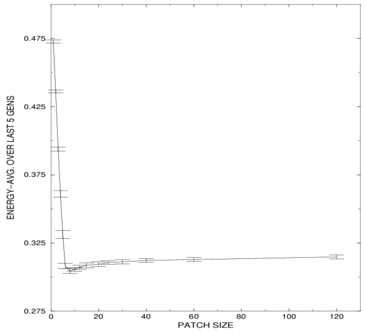
- System partitioned into sub-systems
- Each sub-system ‘selfishly’ optimizes independently of the other sub-systems



Example: Optimizing on subsystems

Kauffman and Macready, Complexity 1995

- Enhanced performance for optimal sub-system size
- The higher the connectivity among decision variables, the smaller the optimal sub-system size



Criticality & Parallelism in Combinatorial Optimization

Macready et al., Science 1996

Increasing parallelism leads to better solutions faster, but up to a degree at which the quality of solutions degrades.

- τ simultaneous local moves (bit flips, *k–opt* exchanges, etc.)
- Optimization on patches, subsystems
- Relaxation of connectivity constraints

Remarks

- *Parallel* = local modifications performed synchronously (i.e., independently). The actual implementation can be sequential.
- No explicit mention to the structure of the system (topology, links between elements, etc.)
- Optimization techniques used are very simple. E.g., gradient descent, simulated annealing.
- A phase transition occurs at the optimal value of *parallelism*.

Criticality & Parallelism in SAT

Questions

- Is this phenomenon involved also in the case of local search applied to the satisfiability problem?
- Under which circumstances does this phenomenon appear?
- Does it appear also when more sophisticated search algorithms are used?
- Is it possible to generalize it?

The Satisfiability problem (SAT)

The problem (model finding): find an assignment to the variables such that the given logical formula is satisfied.

E.g.:

$$\Phi = (a \vee \neg b) \wedge (\neg a \vee c \vee b) \wedge \neg a$$

a solution: $[a, b, c] = [0, 0, 1]$

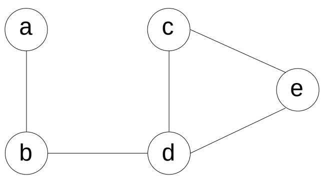
MAXSAT: minimize the number of unsatisfied clauses.

Interaction graph

Rish & Dechter, 1991

$$(a \vee \neg b) \wedge (b \vee d) \wedge (c \vee \neg d \vee \neg e) \wedge (a \vee b)$$

$$\Downarrow$$



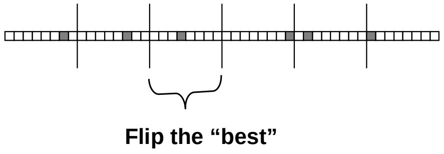
GSAT

Selman et al., AAAI 1992

- Greedy-like algorithm for tackling SAT
- Idea: Flip a variable such that the *score* (i.e., # of clauses unsat $\rightarrow$ sat) is maximal

‘Parallel’ GSAT

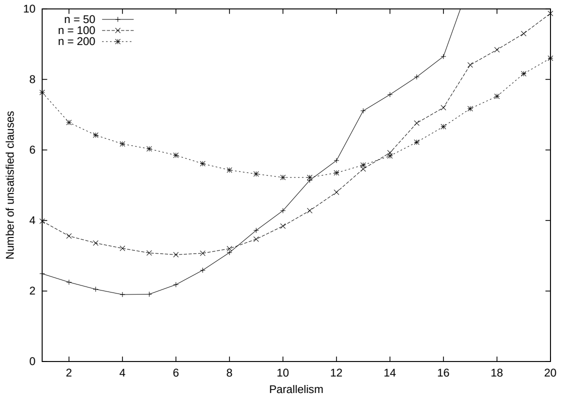
- Divide the set of variables in τ subsets
- Apply a GSAT step in parallel to each subset



Results summary

- Experiments on:
  - Random 3-SAT/MAXSAT instances
  - ‘Structured’ instances from SATLIB
- Optimal sub-set size affected by node degree of interaction graph

Results on random instances

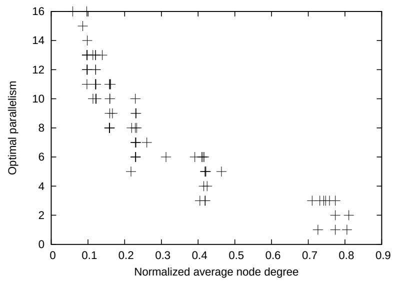


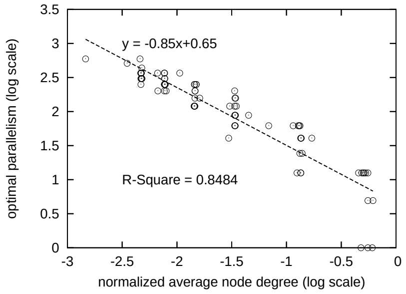
Results on random instances

- τ_{opt} negatively correlated with the *average* node degree of the interaction graph
- The same (normalized) average node degree corresponds to the same value of τ_{opt}, independently of other instance parameters

Results on random instances

A plot from a population of instances



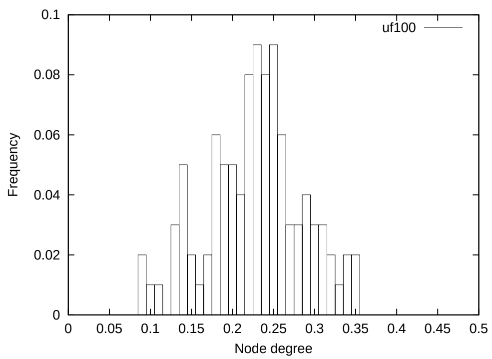


Remarks

- Results are in accordance with previous work by Macready and Kauffman
- The phenomenon is modeled in more general terms by introducing the interaction graph
- The model generalizes previous results on multi-flip local search for SAT

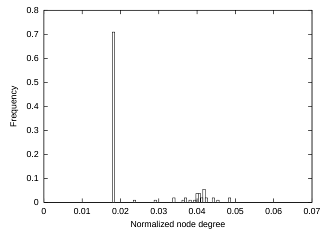
Node degree distribution

- Node degree distribution of 3-SAT/MAXSAT instance interaction graphs are Poissonian ($\sim$Normal)
- Hence average has a strong impact

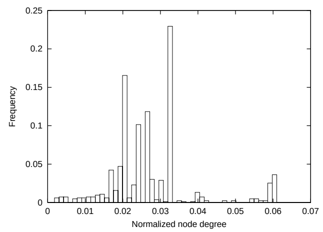


‘Structured’ instances: Node degree frequency

SAT – inductive inference



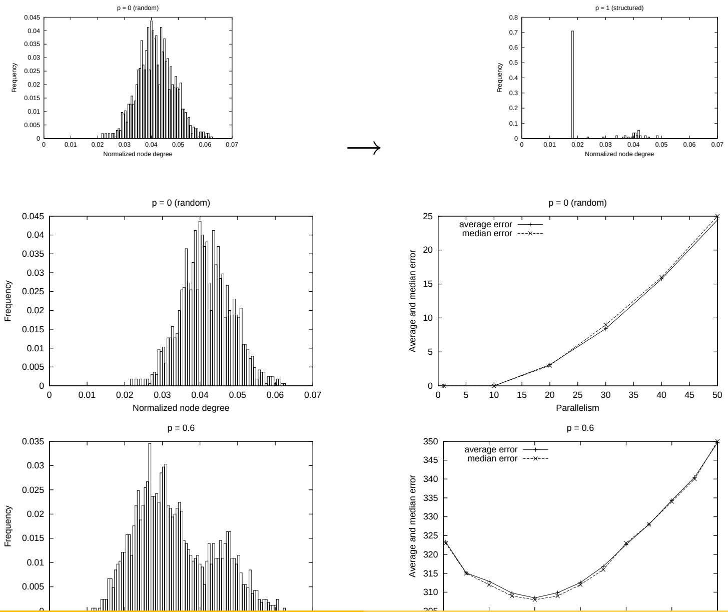
SAT – logistics problem



Results on structured instances

- Same behavior as for random: there exists an optimal value of τ
- **But**: τ_{opt} is affected by the highest peaks (*modes* of the distribution)

Morphing



Discussion and future work

- The phenomenon seems quite general and it can be generalized by modeling system structure as a graph
- Interaction graph is a first approximation: a richer model is required to capture more accurately the interdependence among variables
- A phase transition does not necessarily occur (it depends on the search algorithm)

Discussion and future work

- A successful application: Iterated Local Search for MAXSAT (*Metaheuristic network* european project)
- Different criteria to divide the variables (e.g., based on minimal cuts, adaptive, etc.)
- Extending investigation to different problems and algorithms
- A general model is still missing