Motivation

- Efficient techniques for tackling combinatorial optimization problems exploit the *structure* of the instance to attack
- Strong correlation between search effectiveness and some critical parameters of the instance (e.g., see studies on phase transitions)

Outline

Example: Optimizing on subsystems
Kauffman and Macready, Complexity 1995

- Minimizing the energy of a spin glass system
- Total Energy = \( \sum_i \text{energy}_i \)
Example: Optimizing on subsystems
Kauffman and Macready, Complexity 1995

- System partitioned into sub-systems
- Each sub-system ‘selfishly’ optimizes independently of the other sub-systems

Enhanced performance for optimal sub-system size
- The higher the connectivity among decision variables, the smaller the optimal sub-system size

Criticality & Parallelism in Combinatorial Optimization
Macready et al., Science 1996

Increasing parallelism leads to better solutions faster, but up to a degree at which the quality of solutions degrades.

- \( \tau \) simultaneous local moves (bit flips, \( k\text{-opt} \) exchanges, etc.)
- Optimization on patches, subsystems
- Relaxation of connectivity constraints

Remarks

- \textit{Parallel} = local modifications performed synchronously (i.e., independently). The actual implementation can be sequential.
- No explicit mention to the structure of the system (topology, links between elements, etc.)
- Optimization techniques used are very simple. E.g., gradient descent, simulated annealing.
- A phase transition occurs at the optimal value of parallelism.
Questions
- Is this phenomenon involved also in the case of local search applied to the satisfiability problem?
- Under which circumstances does this phenomenon appear?
- Does it appear also when more sophisticated search algorithms are used?
- Is it possible to generalize it?

The Satisfiability problem (SAT)

The problem (model finding): find an assignment to the variables such that the given logical formula is satisfied.

E.g.:
\[ \Phi = (a \lor \neg b) \land (\neg a \lor c \lor b) \land \neg a \]

A solution: \([a, b, c] = [0, 0, 1]\)

MAXSAT: minimize the number of unsatisfied clauses.

Interaction graph
Rish & Dechter, 1991

\[(a \lor \neg b) \land (b \lor d) \land (c \lor \neg d \lor \neg e) \land (a \lor b)\]

\[\downarrow\]

Greedy-like algorithm for tackling SAT
- Idea: Flip a variable such that the score (i.e., # of clauses unsat \(\rightarrow\) sat) is maximal

GSAT
Selman et al., AAAI 1992

\[\text{Selman et al., AAAI 1992}\]
‘Parallel’ GSAT

- Divide the set of variables in $\tau$ subsets
- Apply a GSAT step in parallel to each subset

Flip the “best”

Results summary

- Experiments on:
  - Random 3-SAT/MAXSAT instances
  - ‘Structured’ instances from SATLIB
- Optimal sub-set size affected by node degree of interaction graph

Results on random instances

- $\tau_{opt}$ negatively correlated with the average node degree of the interaction graph
- The same (normalized) average node degree corresponds to the same value of $\tau_{opt}$, independently of other instance parameters
Results on random instances

A plot from a population of instances

- Results are in accordance with previous work by Macready and Kauffman
- The phenomenon is modeled in more general terms by introducing the interaction graph
- The model generalizes previous results on multi-flip local search for SAT

Remarks

- Node degree distribution of 3-SAT/MAXSAT instance interaction graphs are Poissonian (\sim Normal)
- Hence average has a strong impact

‘Structured’ instances: Node degree frequency
Results on structured instances

- Same behavior as for random: there exists an optimal value of $\tau$
- **But**: $\tau_{\text{opt}}$ is affected by the highest peaks (modes of the distribution)

Discussion and future work

- The phenomenon seems quite general and it can be generalized by modeling system structure as a graph
- Interaction graph is a first approximation: a richer model is required to capture more accurately the interdependence among variables
- A phase transition does not necessarily occur (it depends on the search algorithm)

Morphing

A successful application: Iterated Local Search for MAXSAT (*Metaheuristic network* european project)

- Different criteria to divide the variables (e.g., based on minimal cuts, adaptive, etc.)
- Extending investigation to different problems and algorithms
- A general model is still missing